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COMPARISON OF THREE GENERAL BOUNDS ON THE NUMBER OF MUTUALLY DISJOINT BLOCKS IN INCOMPLETE BLOCK DESIGNS

SANPEI KAGEYAMA*
Department of Mathematics, Hiroshima University, Japan

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Three general bounds on the number of mutually disjoint blocks are known in incomplete block designs. These bounds are totally compared.

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Introduction

There are a large number of bounds on the number of blocks having various block structures. Among them, we can find out several bounds on the number of disjoint blocks for various types of equi-replicated and equiblock-sized block designs. In particular, as general bounds on the number, d, of mutually disjoint blocks in any equi-replicated, equiblock-sized and connected incomplete block design with parameters v, b, r and k, the following three bounds are available (cf. Kageyama [3], Shah [6]) :

$$d \leqq v/k, \tag{1.1}$$

$$d \leqq b - bk(r-1)^2/[\rho(b-r) + k(r^2-b)], \tag{1.2}$$

$$d \leqq b - k(r-1)/B, \tag{1.3}$$

where ρ is the maximum eigenvalue of NN' other than rk for N being the incidence matrix of an incomplete block design and $B = \min\{k, \rho - k + 2(rk-\rho)/b\}$. Bound (1.3) is due to Shah [6]. Of course, these bounds are

*Prepared while visiting Department of Statisties, Haryana Agricultural University, Hissar, India.

also valid for BIB designs and PBIB designs. Note that when the design is a BIB design with parameters v, b, r, k, λ, (1.2) yields the Patwardhan [4] bound $d \leqq (r - k)(r - \lambda)(v - k)/[k(r - k + \lambda(k - 1))]$

if $\rho = r - \lambda$.

The existence of mutually disjoint blocks may provide the possibility of getting a complete replication set which reduces the number of blocks and intra-block variances (cf. Peter [5], similar to statistical properties due to the resolvability of block designs first introduced by Bose [1].

Though Kageyama [3] compared (1.1) with (1.2), in this paper these bounds are totally compared. It is shown that a very simple bound (1.1) is the most stringent in its superiority, provided there are disjoint blocks.

2. Comparisons

We consider an equi-replicated, equiblock-sized and connected incomplete block design with parameters v, b, r and k satisfying a case $B < 0$ only, otherwise (1.3) is meaningless. Note (cf. Kageyama [3]) that $\rho(b - r) + k(r^2 - b) > 0$.

2.1 Comparison of bounds (1.1) and (1.2)

Kageyama [3] has showed the following two lemmas.

LEMMA 2.1. (*i*) *When* $k > \rho$, *(1.2) is more stringent with* (1.1) (*ii*) *When* $k = \rho$, *(1.1) and (1.2) are equivalent.* (*iii*) *When* $k < \rho$, *(1.1) is more stringent than (1.2).*

LEMMA 2.2. *If there are disjoint blocks, then* $k \leqq \rho$.

2.2. Comparison of bounds (1.1) and (1.3)

LEMMA 2.3. *When* $B = k$, *(1.1) is more stringent than (1.3).*

Proof. When $B = k$, it suffices to compare $b - (r - 1)$ and v/k. It follows that

$$\frac{v}{k} < b - r + 1 \Leftrightarrow v < vr - rk + k \Leftrightarrow k(r - 1) < v(r - 1) \Leftrightarrow k < v.$$

The last relation is always valid in any incomplete block design. Hence we have the required result.

LEMMA 2.4. *When* $B = \rho - k + 2(rk - \rho)/b$, (*i*) *if* $r(v - k)/(b - 2) < \rho$, *then (1.3) is more stringent than (1.1)*; (*ii*) *if* $r(v - k)/(b - 2) = \rho$, *then*

(1.1) and (1.3) are equivalent; (iii) if $r(v-k)/(b-2) < \rho$ *then (1.1) is more stringent than (1.3).*

Proof. In this case, it is sufficient to evaluate a difference between v/k and $b - k(r-1)/[\rho - k + 2(rk-\rho)/b]$. It follows that

$$\frac{v}{k} \leqq b - \frac{k(r-1)}{\rho - k + 2(rk-\rho)/b} \Leftrightarrow \frac{bk^2(r-1)}{b(\rho-k) + 2(rk-\rho)} \leqq v(r-1)$$
$$\Leftrightarrow rk \leqq b(\rho-k) + 2(rk-\rho)$$
$$\Leftrightarrow r(v-k)/(b-2) \leqq \rho.$$

Hence we can obtain the required result.

REMARK 1. $B > 0 \Leftrightarrow \rho > r(v-2k)/(b-2)$.

2.3. *Comparison of bounds (1.2) and (1.3)*

LEMMA 2.5. *When* $B = k$, *(1.2) is more stringent than (1.3).*

Proof. In this case, it follows that

$$\frac{bk(r-1)^2}{\rho(b-r) + k(r^2-b)} \geqq r - 1 \Leftrightarrow \rho \leqq rk.$$

Since $\rho < rk$ for a connected design, we have the required result.

LEMMA 2.6. *When* $B = \rho - k + 2(rk-\rho)/b$, *(i) if* $r(v-k)/(b-1) > \rho$, *then (1.3) is more stringent than (1.2); (ii) if* $r(v-k)/(b-1) = \rho$, *then (1.2) and (1.3) are equivalent; (iii) if* $r(v-k)/(b-1) < \rho$, *then (1.2) is more stringent than (1.3).*

Proof. It follows that, since $r \geqq 3$,

$$\frac{1}{B} \leqq \frac{b(r-1)}{\rho(b-r) + k(r^2-b)} \Leftrightarrow \rho \geqq \frac{r(v-k)}{b-1},$$

which yields the required result.

2.4. *Totally Comparisons of Bounds (1.1), (1.2) and (1.3)*

Since

$$\frac{r(v-k)}{b-1} < \frac{r(v-k)}{b-2} < k,$$

it follows from Lemma 2.2 that if there are disjoint blocks, then

$$\frac{r(v-k)}{b-1} < \frac{r(v-k)}{b-2} < k \leqq \rho,$$

which, from Lemmas 2.1, 2.3 to 2.6, yields the following.

PROPOSITION 1. When there are disjoint blocks, a simple bound (1.1) is the most stringent, and (1.2) is more stringent than the Shah bound (1.3).

If one has no information about the existence of disjoint blocks, we can state the following two propositions different from Proposition 1. In some cases, (1.3) also has an opportunity of getting the best position, as in Proposition 3.

PROPOSITION 2. (i) When $\rho \geqq k$, (1.1) is not less stringent than (1.2), which is more stringent than (1.3). (ii) When $\rho \geqq r(v-k)/(b-2)$, bounds (1.1) and (1.2) are not less stringent than (1.3).

REMARK 2. In a BIB design with parameters v, b, r, k, λ, a condition in Proposition 2 (i) becomes "$r \geqq k + \lambda$" which is known to be valid (cf. Kageyama [2], [3]) when the design is resolvable, or k divides v, or the design has disjoint blocks.

PROPOSITION 3. (i) When $\rho \leqq r(v-k)/(b-1)$, (1.3) is not less stringent than (1.2), which is more stringent than (1.1). (ii) When $\rho \leqq r(v-k)/(b-2)$, (1.2) and (1.3) are not less stringent than (1.1).

Note that from Remark 1, we have a restriction on the lower limit of ρ.

REMARK 3. When $B = k$, bounds (1.1) and (1.2) are more stringent than (1.3).

Therefore, when there are disjoint blocks, a trivial bound (1.1) is not less stringent than the other two general bounds (1.2) and (1.3), in an equi-replicated and equiblock-sized incomplete block design, which covers a BIB design and a PBIB design. Of course, if one assumes some additional restrictions on parameters or structures, then there may be the possibility of improving the bound (1.1).

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